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Analytical Solution of ADE with Spatiotemporal Dependence of Dispersion Coefficient and Velocity using Green's Function Method

Abhishek Sanskrityayn[1], Vinod Kumar Bharati[2] and Naveen Kumar[3]

[1,2,3] Department of Mathematics, Institute of Science, Banaras Hindu University
Varanasi-221005, India
[Email: abhi.bhu2008@gmail.com; vinodkmr418@gmail.com; navkumar50s@gmail.com]

Abstract

One dimensional advection diffusion equation (ADE) with variable coefficients is solved analytically using Green's Function Method (GFM). ADE describes the pollutant mass transport in a heterogeneous medium originating from an instantaneous source. The heterogeneity of the medium is delineated through considering dispersion coefficient and velocity as functions of both the independent variables. The source term is defined by the non-homogeneous term in the ADE. A moving coordinate transformation is developed through which variable coefficients of the ADE are reduced into constant coefficients.

Key words: Solute transport, ADE, Variable coefficients, Heterogeneity, Green's Function Method

1. Introduction

Pollutants mass transport in a porous medium like aquifer and oil reservoirs or in an open medium like air and river is described by a parabolic type of partial differential equation known as Advection Diffusion Equation (ADE). It may be derived on the basis of the principle of conservation of mass. In one dimension it has mainly two coefficients: Dispersion coefficient (D) and mean velocity of the medium (u). The one very effective way to describe solute transport in a medium is through the analytical and numerical solutions of advection-diffusion equation (ADE) associated with an initial and boundary conditions. Solute transport though homogeneous and isotropic medium have been described by ADE with constant coefficients and analytical solutions of such problems have been compiled by (Van Genuchten and Alves, 1982). In real scenario most natural environments are heterogeneous. On the basis of field evidence, experiment studies and statistical approach many authors (Matheron and de Marsily, 1980; Sposito et al., 1986; Güven at el., 1984) have suggested that due to the heterogeneity, the porosity of a porous medium or ability to transport through a non-porous medium varies with position and time hence velocity of the flow through the medium also depends upon position and time variable. Many research papers have used this theory to describe the solute transport in heterogeneous media through analytical and numerical solutions (Yates, 1990; Basha and El-Habel, 1993; Atul et. al., 2010). Analytical solutions for solute transport problems using GFM was presented by Leij et al. (2000), in which solutions for multidimensional problems may be obtained by multiplying the Green's function that can be used to formulate the solution for a wide variety of transport problems in infinite, semi-infinite, and finite media. Leij and Van Genuchten (2000) derived the specific solutions for transport from a rectangular source (parallel to the flow direction) of persistent contamination using first-, second-, or third-type boundary or source input conditions using GFM. Park and Zhan (2001) solved the general form of contaminant transport from three-dimensional finite, instantaneous and continuous sources in a finite-thickness aquifer, using GFM and derived the solutions for point, line, and area sources. Singh et al. (2009) have obtained the analytical solution of the one-dimensional ADE with sinusoidally varying and exponentially decreasing time dependent velocity. Gao et al. (2012) solved the solute transport problem through heterogeneous finite media using a combined mobile-immobile model, asymptotic dispersivity function of travel distance and Laplace transformation technique and extended power series method. Van Genuchten et al. (2013) presented a series of one- and multi-dimensional solutions of the standard equilibrium advection diffusion equation with and without terms accounting for zero order production and first order decay. You and Zhan (2013) derived solutions for solute transport in one dimensional finite domains with distance dependent dispersion coefficient and time dependent source and compared them with the corresponding solutions for semi-infinite domain to investigate the effects of outer boundary conditions. Singh et al. (2013) obtained analytical solution of one-dimensional solute transport with transient groundwater flow in semi-infinite and homogeneous medium for a temporally dependent pulse type source introduced at an intermediate position of the aquifer, using the Laplace Transform Technique (LTT). Suk (2013) presented a semi-analytical solution for multispecies transport coupled with a sequential first-order reaction network under variable flow velocities and dispersion coefficients by employing the generalized integral transform technique and the general linear transformation method. Zamani and Bombardelli (2014) have obtained the analytical solutions of nonlinear advection diffusion equation with space and time dependent dispersion coefficient and velocity. Mahato et al. (2015) discussed the two-dimensional solute dispersion in saturated homogeneous and anisotropic porous media and solved analytically, using the Laplace Transform Technique (LTT) and numerically with the help of Explicit Finite Difference (EFD) method.

In the present paper heterogeneity of the medium is described by variable coefficients of the ADE. Both coefficients dispersion coefficient and velocity are considered functions of both the independent variables. Different expressions for both the parameters are considered. A coordinate transformation equation is developed which reduces the variable coefficients of ADE into the constant coefficients. Then Green' function method is used to obtain the analytical solution. The effect of temporal dependence of both parameters on the concentration pattern is studied. Thus the present work obtains the analytical solution of an ADE with both its parameters being functions of both the independent variables for the instantaneous source, which has been elusive so far. Such solution may be very close in predicting the concentration pattern in real scenario where pollution material is injected instantly in a medium.

2. Mathematical Formulation and Analytical Solution

Let us consider linear advection-diffusion equation (ADE) in one dimension in general form as

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left[D_0 f_1(x,t)\frac{\partial c}{\partial x} - u_0 f_2(x,t)c \right] + q(x,t)\,, \tag{1}$$

where c is the solute concentration at a position x at time t, $q(x,t)$ represents the instantaneous source term for the pollutant mass in the infinite domain, dispersion coefficient $D(x,t)$ and velocity $u(x,t)$ through the medium are written as $D_0 f_1(x,t)$ and $u_0 f_2(x,t)$, respectively, and which will be D_0 and u_0, respectively in a homogeneous medium. Equation (1) may be written as

$$\frac{\partial c}{\partial t} = D_0 f_1 \frac{\partial^2 c}{\partial x^2} + D_0 \frac{\partial f_1}{\partial x}\frac{\partial c}{\partial x} - u_0 f_2 \frac{\partial c}{\partial x} - u_0 \frac{\partial f_2}{\partial x} c + q(x,t) \tag{2}$$

We wish to reduce above equation in the form

$$\frac{\partial C}{\partial t'}=D_0 f^2(t')\frac{\partial^2 C}{\partial X^2}-\alpha C+q_1(X,t'), \tag{3}$$

where α is a parameter of dimension inverse of time variable. Its analytical solution is known (Basha and El-Habel, 1993). To get it in terms new independent variables following set of transformation equations is used:

$$X=X(x,t),\ t=t' \tag{4}$$

The equation (3) is dimensionally correct as $f(t)$ is dimensionless expressions, as evident in the later part of the paper. ADE in equation (2) may be written as (in place of new time variable t' we are using t)

$$\frac{\partial C}{\partial t}=D_0 f_1\left(\frac{\partial X}{\partial x}\right)^2\frac{\partial^2 C}{\partial X^2}+\left[D_0\frac{\partial}{\partial x}\left(f_1\frac{\partial X}{\partial x}\right)-u_0 f_2\frac{\partial X}{\partial x}-\frac{\partial X}{\partial t}\right]\frac{\partial C}{\partial X}-u_0\frac{\partial f_2}{\partial x}C+q_1(x,t) \tag{5}$$

If equation (5) is same as equation (3) then

$$f_1\left(\frac{\partial X}{\partial x}\right)^2=f^2, \tag{6a}$$

$$D_0\frac{\partial}{\partial x}\left(f_1\frac{\partial X}{\partial x}\right)-u_0 f_2\frac{\partial X}{\partial x}-\frac{\partial X}{\partial t}=0\quad, \tag{6b}$$

and $$\frac{\partial f_2}{\partial x}=\frac{\alpha}{u_0} \tag{6c}$$

Solving (6c), we get expression of $f_2(x,t)$, hence the expression for velocity may be obtained as

$$u(x,t)=u_0 f_2(x,t)=\alpha x+u_0\phi(t) \tag{7}$$

Using equations (6a), and (7) in equation (6b) we have

$$\frac{D_0 f}{2\sqrt{f_1}}\frac{\partial f_1}{\partial x}-\left(\alpha x+u_0\phi(t)\right)\frac{f}{\sqrt{f_1}}-\frac{\partial X}{\partial t}=0 \tag{8}$$

Integrating equation (6a) with respect to x, we have

$$X=\int\frac{f dx}{\sqrt{f_1}} \tag{9}$$

Using it in equation (8) we get

$$\frac{D_0 f}{2\sqrt{f_1}}\frac{\partial f_1}{\partial x}-\left(\alpha x+u_0\phi(t)\right)\frac{f}{\sqrt{f_1}}=\frac{\partial}{\partial t}\left(\int\frac{f dx}{\sqrt{f_1}}\right) \tag{10}$$

Now we choose an expression for $f_1(x,t)$ as

$$f_1(x,t)=ax+1/(1+mt), \tag{11}$$

where a and m are spatial and temporal dependence parameters, respectively. Both are of the dimensions such that ax and mt are dimensionless terms. Equation (10) will become

$$\frac{aD_0 f}{2\sqrt{ax+1/(1+mt)}}-\frac{(\alpha x+u_0\phi)f}{\sqrt{ax+1/(1+mt)}}=\frac{\partial}{\partial t}\left(\int\frac{f dx}{\sqrt{ax+1/(1+mt)}}\right)$$

or $$\frac{aD_0 f-2u_0 f\phi-2\alpha fx}{2\sqrt{ax+1/(1+mt)}}=\frac{2x\dfrac{df}{dt}}{\sqrt{ax+1/(1+mt)}}+\frac{\dfrac{2}{a}\dfrac{df}{dt}\left[1/(1+mt)\right]-\dfrac{mf}{a}\left[1/(1+mt)^2\right]}{\sqrt{ax+1/(1+mt)}} \tag{12}$$

Above equation holds good if

$$2\frac{df}{dt}=-\alpha f \tag{13a}$$

and

$$\frac{2[1/(1+mt)]}{a}\frac{df}{dt}-\frac{mf}{a(1+mt)^2}=\frac{aD_0 f}{2}-u_0 f(t)\phi(t) \tag{13b}$$

From equation (13a) we get expression of $f(t)$ as

$$f^2(t)=\exp(-\alpha t) \tag{14}$$

Using equation (7), (13a) and (13b) we may get the expression for $\phi(t)$ and so velocity may finally be written as

$$u(x,t)=u_0 f_2(x,t)=\left[\alpha x+\frac{\alpha}{a(1+mt)}+\frac{m}{a(1+mt)^2}+\frac{aD_0}{2}\right] \tag{15}$$

Using equations (15) for $f_2(x,t)$, and equation (11) for $f_1(x,t)$, ADE (1) may be considered as

$$\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left[D_0\left\{ax+\frac{1}{1+mt}\right\}\frac{\partial c}{\partial x}-\left\{\frac{1}{2}aD_0+\alpha x+\frac{\alpha}{a(1+mt)}+\frac{m}{a(1+mt)^2}\right\}c\right]+q(x,t);\ -\infty<x<\infty,\ t>0 \tag{16}$$

An initial condition is assumed as

$$c(x,t=0)=C_i\omega(x);\quad -\infty<x<\infty \tag{17}$$

Equation (16) may be reduced into the form

$$\frac{\partial C}{\partial t}=D_0\exp(-\alpha t)\frac{\partial^2 C}{\partial X^2}-\alpha C+q_1(X,t), \tag{18}$$

by using the transformation (obtained by using equations (9), (11) and (14))

$$X^2=\frac{4\exp(-\alpha t)}{a^2}\left|ax+\frac{1}{1+mt}\right| \tag{19}$$

The decay term in equation (18) is eliminated by introducing a new dependent variable through the transformation

$$K(X,t)=C(X,t)\exp(\alpha t) \tag{20}$$

Further using the transformation introducing a new time variable

$$T=\int_0^t \exp(-\alpha v)dv, \tag{21}$$

we may finally get the diffusion equation with constant coefficient and non-homogeneous term as

$$\frac{\partial K}{\partial T}=D_0\frac{\partial^2 K}{\partial X^2}+Q(X,T)\exp(2\alpha t), \tag{22}$$

where exponential expression in old time may be converted in new time variable using equation (21). The initial condition may assume the form as

$$K(X,T=0)=C_i\omega\left[\frac{aX^2}{4}-\frac{1}{a}\right];\qquad -\infty<X<\infty \tag{23}$$

The analytical solution of diffusion equation in equation (22) with initial condition in equation (23) may be obtained by Green's Function Method (Habermann, 1987) as

$$K(X,T)=\int_0^T\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_0(T-\zeta)}}\exp\left(-\frac{(X-\chi)^2}{4D_0(T-\zeta)}\right)^2 Q(\chi,\zeta)\exp(2\alpha\zeta)d\chi d\zeta$$

$$+C_i\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_0 T}}\exp\left(-\frac{(X-\chi)^2}{4D_0 T}\right)\omega\left[\frac{a\chi^2}{4}-\frac{1}{a}\right]d\chi, \tag{24}$$

where χ and ζ are the dummy variables. Using transformation equations (21), (20) and (19) one by one the analytical solution in original independent variables may be obtained as

$$c(x,t)=\int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_0(T-\zeta)}}\exp\left(-\frac{(\eta-\chi)^2}{4D_0(T-\zeta)}\right)Q(\chi,t_0)\exp\{2\alpha(t-t_0)\}\,d\chi dt_0$$
$$+C_i\exp(-\alpha t)\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_0T}}\exp\left(-\frac{(X-\chi)^2}{4D_0T}\right)\omega\left[\frac{a\chi^2}{4}-\frac{1}{a}\right]d\chi \qquad (25)$$

Let us assume

$$\xi=a\chi^2\exp(\alpha t_0)/4 \qquad (26)$$

Then analytical solution in equation (25) may be written as

$$c(x,t)=\int_0^t\int_0^{\infty}\frac{1}{\sqrt{4\pi D_0(T-\zeta)}}\exp\left(-\frac{(X-\eta')^2}{4D_0(T-\zeta)}\right)\frac{q\left[\xi-1/a(1+mt_0),t_0\right]}{\exp(\alpha t)}\sqrt{\frac{\exp(\alpha t_0)}{a\xi}}\,d\xi dt_0$$
$$+\frac{C_i\exp(-\alpha t)}{\sqrt{4\pi D_0T}}\int_{-\infty}^{\infty}\exp\left(-\frac{(X-\chi)^2}{4D_0T}\right)\omega\left[\frac{a\chi^2}{4}-\frac{1}{a}\right]d\chi\,; \qquad (27)$$

where

$$\eta'=2\sqrt{\frac{\xi}{a\exp(\alpha t_0)}} \qquad (28)$$

$$\zeta=\int_0^{t_0}\exp(-\alpha v)dv \qquad (29)$$

2.1 Solutions for instantaneous point injection with zero initial concentration distribution

For simplicity let us assume initially solute free medium that is let $C_i=0$. The instantaneous non-dimensional injection of a tracer may be defined as

$$q(x,t)=M\delta(x)\delta(t) \qquad (30)$$

where M is the non-dimensional mass injected and $\delta(x)$ is the Dirac delta function. From analytical solution (27) we get the solution as

$$c(x,t)=\frac{M}{\exp(\alpha t)}\frac{1}{\sqrt{4\pi D_0T}}\exp\left[-\frac{1}{4a^2D_0T}\left(2\sqrt{\frac{ax+1/(1+mt)}{\exp(\alpha t)}}-2\right)^2\right] \qquad (31)$$

3. Discussion and conclusion

The analytical solution obtained in equation (31) is illustrated in figure 1 in a longitudinal domain $0\le x(\text{meter})\le 100$. The input values are chosen as: injected mass $M=1.0$, reference dispersion coefficient and velocity in homogeneous medium $D_0(\text{m}^2/\text{hr})=0.2$, and $u_0(\text{m/hr})=0.25$, respectively, the first order decay term coefficient $\alpha(\text{hr}^{-1})=0.025$, the spatial dependence parameter $a(\text{m}^{-1})=0.1$, and two values of temporal dependence parameter $m(\text{hr}^{-1})=0.05$ and 0.1. The figure 1 is drawn for at $t(\text{hr})=5$, 25 and 50. Dotted line curves are drawn for the higher value of m while solid curves are drawn for the lower value $m(\text{hr}^{-1})=0.05$. It is evident that the peak concentration occurs near the source and it lowers down and drifts away from the source with time. The concentration of the solute mass to be dispersed that is the concentration level at the source location decreases very fast with time, an essential property of an instantaneous source. Concentration level at a position and time depends upon the temporal dependence of dispersion coefficient and velocity. From the expressions of the two occurring in the ADE in equation(16) it may be evaluated that at a position and at each time dispersion coefficient decreases but velocity increases with temporal dependence parameter *m*. However, velocity starts decreasing with *m* after some time t = 20 (hr). In other words advection dominates dispersive property of the pollutant mass as a result effect of temporal dependence reduces from this time as compared to that at t = 5 (hr). Though by this time in case of an instantaneous source concentration level becomes very low in a sizable portion of the domain away from the injected location yet the domination of advection over dispersive ability of the pollutant mass ensures faster recovery of a polluted domain. For higher value of temporal dependence parameter, *m* the dispersion will have lower parameter value hence in this case the peak will be nearer to the instantaneous source location than that in case of lower value of *m*. Dispersion coefficient and velocity decrease with time at a particular value m = 0.05 or 0.1 $(hr)^{-1}$, resulting in decreasing solute transport through the medium. It may be evident from the figure that peak concentration does not come down from 25 (hr) to 50 (hr) at the same rate as from 5 (hr) to 25 (hr).

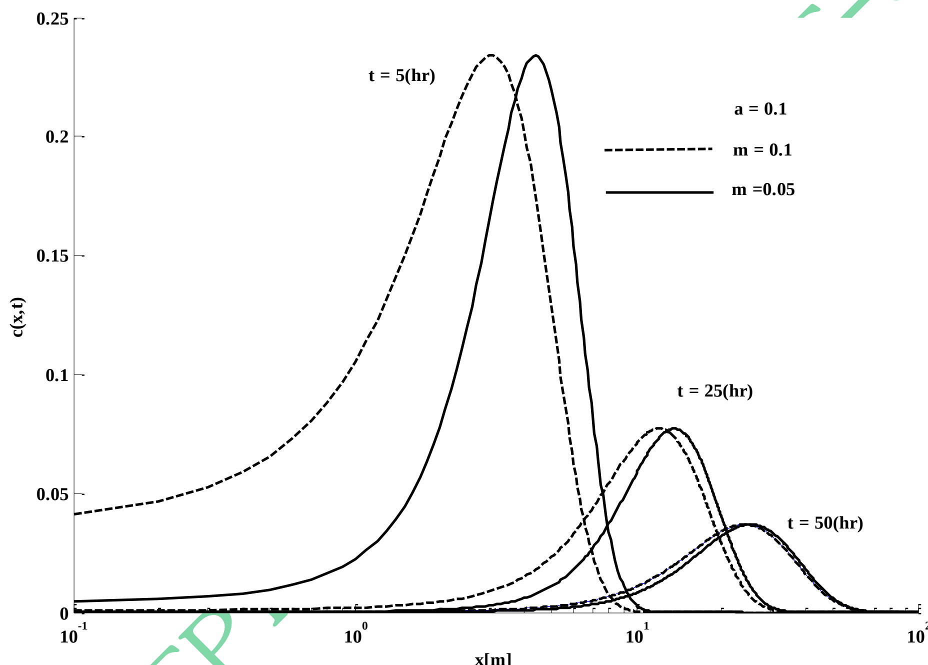

Fig.1. Concentration distribution from instantaneous injection source along heterogeneous medium.

Thus the assumption of dispersion coefficient and velocity as function of both the independent variables describing the heterogeneity of the medium in closer way than that in the previous papers, affects the concentration level in a finite domain due to an instantaneous source in a significant way. But it makes the ADE difficult to solve analytically. Solution is obtained for an instantaneous source. The temporal dependence of dispersion coefficient and velocity is considered to be decelerating while spatial dependence is considered to be of increasing trend. But in this type of source concentration level comes down very fast as we move away from the source location so this increasing dependence does not affect much. It is important to note that the system of advection – diffusion equation reduced in the form as in equation (22) along with initial condition in equation (23) may also be solved using Laplace integral transform and Fourier integral transform techniques but the type of expressions for dispersion coefficient and velocity considered in this paper prohibits its. So Green's function method remains the only option to get the analytical solution. But to use this method too, the ADE with variable coefficients needs to be reduced into a form with constant coefficients. It has been accomplished through a pertinently developed coordinate transformation.

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Corresponding Author

Abhishek Sanskrityayn

Department of Mathematics, Institute of Science, Banaras Hindu University

Varanasi-221005, India

Email: abhi.bhu2008@gmail.com

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