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Analytical Solution of ADE with Linear Spatial Dependence of Dispersion Coefficient and Velocity using GITT

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Abstract

Analytical solution of advection diffusion equation (ADE) with variable coefficients is obtained in the present paper. The dispersion coefficient and velocity are proportional to non-homogeneous linear expression in position variable. So far no analytical solution of the ADE with such coefficients is available. In the existing literature the solutions of the ADE with spatially dependent coefficients of linear or quadratic form using GITT are also not purely analytical. But in the present paper it has been possible because of choosing a SLP different from that considered in the previous works using which the Fourier series convergences to the concentration attenuating pattern with much less number of terms. The series produces almost same result for three, five and eight first terms. The concentration values evaluated from the analytical solution matches exactly with those obtained from numerical solutions of the same problem. The domain is used finite so that GITT may be used. The source of pollutant's solute mass dispersing in a medium is assumed uniform and continuous and is expressed by a non-homogeneous first type condition at the origin and second boundary is assumed permeable expressed by a non-homogeneous flux type condition. The concentration pattern described by the analytical solutions along with their convergence is illustrated.

Keywords: Sturm-Liouville's problem (SLP); Eigen function; Orthonormal function; Heterogeneity.

1. Introduction

Pollutants dispersion in a medium of air, soil and water is described a parabolic type partial differential equation known as advection diffusion equation (ADE). Pollutants concentration level at a position away from its source at a time may be estimated through analytical and numerical solutions of the ADE subjected to initial and boundary conditions. Heterogeneity of the medium and unsteadiness of its properties make difficult to integrate the ADE analytically. Sometimes nature and type of the pollutants source constricts using the analytical methods. With the advent of variety of software packages more and more numerical solutions are coming but they lack the reliability. So the analytical solution in realistic situation is the necessity to assess and manage the environmental pollution occurring due to variety of human activities in true sense. The two important parameters of the ADE influencing the pollutants concentration dispersion in a medium are dispersion coefficient and velocity. Heterogeneity of the medium in which transport properties like permeability, hydraulic conductivity vary with position is delineated by spatial/ temporal dependence of dispersion coefficient/ velocity (Pickens and Grisak, 1981; Dagan, 1987). Dispersion theories for mass concentration dispersion in open or porous media have been established according to which dispersion coefficient is proportional to nth power of velocity where the index n varies between 1 and 2 (Freeze and Cherry, 1979; Scheidegger, 1957 and; Rumer, 1962). In these two situations ADE remains linear but its two coefficients become variable. To solve such an ADE using an integral transform technique the variable coefficients need to be reduced into constant coefficients. A number of analytical solutions of the ADE occur in which variety of spatial or temporal dependent coefficients are considered but so far no analytical solution exists in case the dispersion coefficient and velocity are proportional to linear non-homogeneous expression in position variable. Very recently a semi-analytical solution has been obtained (Jia et al., 2013). In the present paper analytical solution of the ADE with such variable coefficients has been obtained using generalized integral transform technique (GITT).

Though ADE with variable coefficients of linear or quadratic spatial expressions has been solved using this method (GITT) but they are also not purely analytical solutions. This technique is used in a finite domain. An auxiliary system of equations known as Sturm-Liouville Problem (SLP) comprising of homogeneous linear second order ordinary differential equation (ODE) with self-adjoint operator along with two homogeneous boundary conditions are chosen. The non-zero solutions of this problem are referred to as eigen functions which are orthogonal to each other. The solution of the ADE is assumed in the form of an extended Fourier series in terms of these eigen functions with time dependent coefficients. Using this solution, the ADE along with the initial condition is reduced into a system of first order ordinary differential equations in the time dependent coefficients of the series along with the initial conditions. This initial value problem is solved either by matrix method or by using Laplace transformation technique. Liu et al. (2000) have used the hybrid numerical-analytical approach GITT (Cotta, 1993, 1998) to solve the one-dimensional advection-dispersion equation coupled with linear/ nonlinear sorption and decay, and spatially and temporally variable flow and dispersion coefficient. Cotta et al. (2003) proposed a lumped-differential formulation of the porous matrix diffusion phenomenon, based on the coupled integral equations approach to study the contaminant transport within the fractured porous media, and the system of two coupled partial differential equations is solved using GITT. Wortmann et al. (2005) presented an analytical solution of the ADE with constant eddy diffusivity coefficient and wind speed profile to simulate the pollutant dispersion in the planetary boundary layer using GITT. This work was extended by Moreira et al. (2005) by using the same method utilizing an eddy diffusivity depending upon source distance. Costa et al. (2006) presented a three dimensional solution of the ADE considering a vertically inhomogeneous planetary boundary layer using GITT. Naveira et al. (2007) used the GITT to get the hybrid numerical-analytical solution of transient laminar forced convection over flat plates, subjected to arbitrary time variations of wall heat flux applied from above. Vilhena et al. (2008) presented a semi-analytical solution the three-dimensional steady-state ADE, considering non-local turbulence closure using generalized integral advection diffusion multilayer technique, an extended method of GITT. Cassol et al. (2009) obtained analytical solution for the transient two-dimensional atmospheric pollutant dispersion problem utilizing the double GITT, the Laplace transformation and the matrix diagonalization. Cotta et al. (2010) employed GITT to yield a hybrid numerical-analytical solution of the bio-heat model in heterogeneous media. Chen et al. (2011) presented exact analytical solutions to the two-dimensional ADE in cylindrical coordinates in finite domain subject to the first- and third-type inlet boundary conditions, using the second kind finite Hankel transform and GITT. Guerrero et al. (2013) presented an analytical solution of ADE with first order decay term for multi-layered media using the classical integral transform technique (CITT), in which the solution procedure used an associated non-self-adjoint advection-diffusion eigen value problem.

But so far the system of initial value problem has been solved numerically in all the previous works using GITT because the extended Fourier series converges to the concentration attenuation pattern for large number of terms. In the present paper the dispersion coefficient and velocity are considered proportional to each other and both have same linear non-homogeneous expression in position variable. This spatial variability of the two parameters owe to the heterogeneity of the medium. The domain is considered finite. The pollutant source is considered uniform and continuous and is formulated by a solution type non-homogeneous condition at the origin of the domain. The extreme boundary is considered permeable expressed by a non-homogeneous flux type boundary condition. To use GITT, an ODE different from that considered in the previous works is chosen as part of the Sturm-Liouville's problem (SLP). As a result the extended Fourier series converges to the concentration distribution pattern originating from a uniform continuous source for very less number of terms. Hence it has been possible to write the analytical solutions. The analytical solutions are written for three, five and eight first terms of the series. All these solutions are illustrated from which the convergence and stability of the analytical solutions are very much evident. The concentration values evaluated from the analytical solution for five terms matches exactly with those obtained numerically at each time. It validates the analytical solution. It is also shown that the results obtained by using the ODE which is same in all the SLPs of the previous works using GITT are far from convergence with same number of terms.

2. Mathematical Equations and Analytical Solutions using GITT

A one-dimensional advection diffusion equation with only diffusive and advective terms in general form is

$$\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left[D(x,t)\frac{\partial c}{\partial x}-u(x,t)c(x,t)\right], \tag{1}$$

where $c(x,t)$ is the pollutant concentration at position x of the medium at time t and the two coefficients $D(x,t)$ and $u(x,t)$ are dispersion coefficient and the advective velocity, respectively. Both are considered proportional to each other. As the medium is considered heterogeneous so each of both the parameters is considered linear non-homogeneous expression in position variable.The ADE in equation (1) may be written as

$$\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left[D_0\left(1+ax\right)\frac{\partial c}{\partial x}-u_0\left(1+ax\right)c(x,t)\right], \tag{2}$$

where D_0 and u_0 are dispersion coefficient and velocity in homogeneous medium, respectively and a is the spatial dependence parameter of dimension inverse of that of x . Initially the domain is considered solute free that is

$$c(x,t=0)=0\,;\ 0\le x\le l \tag{3}$$

The origin of the domain is considered at the location of uniform continuous pollution point source. It is referred to an input concentration. It is expressed as

$$c(x=0,t)=\beta c_0\,;\ t>0 \tag{4}$$

The other end of the domain is considered permeable and is expressed as

$$\frac{\partial c}{\partial x}(x=l,t)=\gamma c_0\,;\ t>0 \tag{5}$$

Using the non-dimensional variables

$$C^*=c/c_0,\ X=x/l\,, T=D_0t/l^2,\ U=u_0l/D_0,\ b=al \tag{6}$$

pollutants dispersion problem expressed by equations (2) – (5) assume the form

$$\frac{\partial C^*}{\partial T}=(1+bx)\frac{\partial^2C^*}{\partial X^2}+\left\{b-U(1+bX)\right\}\frac{\partial C^*}{\partial X}-UbC^*, \tag{7}$$

$$C^*(X,T=0)=0\,;\ 0\le X\le 1, \tag{8}$$

$$C^*(X=0,T)=\beta\,;\quad T>0, \tag{9}$$

$$\frac{\partial C^*}{\partial X}(X=1,T)=\gamma\,;\ \ T>0, \tag{10}$$

To consider a SLP equivalent to the dispersion problem given by equations (7) – (10) the two boundary conditions are made homogeneous through the following transformation

$$C^{*}(X,T)=C(X,T)+\beta+\gamma X\ , \tag{11}$$

Equations (7) – (10) become

$$\frac{\partial C}{\partial T}=(1+bx)\frac{\partial^2 C}{\partial X^2}+(b-U-UbX)\frac{\partial C}{\partial X}-UbC+(b-U-2Ub)\gamma-Ub\beta \tag{12}$$

$$C(X,T=0)=-(\beta+\gamma X)\ ;\ 0\le X\le 1, \tag{13}$$

$$C(X=0,T)=0\ ;\quad T>0, \tag{14}$$

$$\frac{\partial C}{\partial X}(X=1,T)=0\ ;\quad T>0 \tag{15}$$

To solve this dispersion problem using GITT, following SLP is considered

$$\frac{d^2 y}{dX^2}+\frac{dy}{dX}+(1+\lambda)\,y=0\ , \tag{16}$$

$$y(0)=0 \tag{17}$$

and

$$\left[\frac{dy}{dX}\right]_{X=1}=0 \tag{18}$$

The operator of the ordinary differential equation (16) may be shown self-adjoint by multiplying it by a term $\exp(X)$. Whereas in the previous works following ODE has been used

$$\frac{d^2 y}{dX^2}+\lambda^2 y=0 \tag{19}$$

The trivial solution of the problem given by Equations (16) – (18) is $y=0$.The non-trivial solutions are called the eigen functions, and may be obtained as (Haberman 1987; Kreyszig 2011)

$$\phi_n(X)=\exp(-X/2)\sin(\alpha_n X)\ , \tag{20}$$

where $\alpha_n=\left(\sqrt{3+4\lambda_n}\right)/2$, and α_n are the roots of $\tan\alpha=2\alpha$, $n=1,2,3,\ldots$. These functions are orthogonal to each other. Let $\psi_n(X)$ be orthonormal function corresponding to the eigen function $\phi_n(x)$, defined as

$$\psi_n(X)=\frac{\phi_n(X)}{\|\phi_n(X)\|}\ , \tag{21}$$

where $\|\phi_n(X)\|^2=\int_{x=0}^{1}\phi_n^2(X)dX$ is the square norm of function $\phi_n(X)$. Hence orthonormal functions are also orthogonal to each other., *i.e.*,

$$\int_0^1 \exp(X)\psi_n(X)\psi_m(X)dX=\begin{cases}1, n=m\\ 0, n\neq m\end{cases} \tag{22}$$

Let us consider the solution of the dispersion problem in the form of extended Fourier series as

$$C(X,T)=\sum_{n=1}^{m\to\infty}c_n(T)\,\psi_n(X)\ , \tag{23}$$

where $c_1,c_2,c_3,\ldots$ are the time dependent coefficients to be determined. To consider the convergence of this series finite values of m are chosen. Substituting this solution in equation (12) and equation (13), multiplying both equations by $\exp(X)\psi_p(X)$, integrating over the given domain, and using the orthogonality property given in equation (22), we may get a system of first order ordinary differential equations, in matrix form, as

$$A\frac{d\,c_n(T)}{dT}+B\,c_n(T)=f(X),\quad n=1,2,3....,\tag{24}$$

with the system of initial condition as

$$c_n(0)=-\frac{(\beta+\gamma X)}{N_n^{1/2}}\left\{\int_0^1 \exp(X/2)\sin(\alpha_n X)\,dX\right\},\tag{25}$$

where, $A=\frac{1}{N_n^{1/2}N_p^{1/2}}\{a_{n,p}\}$ and $B=\frac{1}{N_n^{1/2}N_p^{1/2}}\left[\{b_{n,p}\}+\{c_{n,p}\}+\{d_{n,p}\}+\{e_{n,p}\}\right]$ are matrices of order $n\times p$, and $f=\frac{1}{N_p^{1/2}}\left[\{f_{1n}\}+\{f_{2n}\}\right]$ is a matrix of order $n\times 1$, and $N_n^{1/2}$ and $N_p^{1/2}$ are n^th^ and p^th^ norm of orthogonal function. The elements of these matrices are as follows:

$$a_{n,p}=\int_0^1 \sin(\alpha_n X)\sin(\alpha_p X)dX$$

$$b_{n,p}=(\alpha_n^2-0.25+0.5(b-U)+Ub)\int_0^1 \sin(\alpha_n X)\sin(\alpha_p X)dX$$

$$c_{n,p}=(1-b+U)\alpha_n\int_0^1 \cos(\alpha_n X)\sin(\alpha_p X)dX$$

$$d_{n,p}=(b(\alpha_n-0.25)+0.5Ub)\int_0^1 X\sin(\alpha_n X)\sin(\alpha_p X)dX$$

$$e_{n,p}=b\alpha_n(1+U)\int_0^1 X\cos(\alpha_n X)\sin(\alpha_p X)dX$$

$$f_{1n}=(\gamma(b-U)-Ub)\int_0^1 \exp(X/2)\sin(\alpha_n X)\,dX$$

and $$f_{2n}=-2Ub\gamma\int_0^1 X\exp(X/2)\sin(\alpha_n X)\,dX$$

where n and p vary from 1 to m, the number of terms in the series in equation (23). The value of m decides the order of each matrix. It is clear that the matrix A is an identity matrix and matrix B may be decomposed as

$$B=VDV^{-1},\tag{26}$$

where D is the diagonal matrix of the eigen values of the matrix B, and the eigen vectors corresponding to each eigen-value form the matrix V. Taking the Laplace integral transform of equation (24) and using equation (25) in that process and then using equation (26) we have

$$c_n(s)=V(sI+D)^{-1}V^{-1}\{c_n(0)+L(f)\}\tag{27}$$

where s is the parameter and L is the operator of the Laplace transform. Taking the inverse Laplace transform of equation (27), the column matrix $c_n(T)$ is obtained, the elements of which are the desired coefficients of the extended Fourier series in equation (23). With the help of these coefficients and the orthonormal functions from equation (21) the solution $C(X,T)$ may be written. Further using the transformation equation (11) and the non-dimensional variables the desired solution $c(x,t)$ may be obtained.

3. Results and Discussion

The concentration values are evaluated in the domain $0 \le x(\text{km}) \le l$, where $l = 1$ is considered. Initially the domain is considered solute free. More input values are considered as: velocity and dispersion coefficient in homogeneous medium $u_0 = 0.1$ (km/year) and $D_0 = 0.14$ (km^2/year), respectively and spatial dependence parameter $a = 0.1$ (year)$^{-1}$. The two constants occurring in equations (4) and (5) are taken as $\beta = 1$ and $\gamma = 0.05$, respectively. Also $c_0 = 1.0$ is taken. Following three cases are discussed with different number of terms, m in equation (23). It may be noted that the eigen values of matrix B changes due to change in any one of the above input values and also with the different number of terms m but they will remain real unrepeated non-zero values.

Case 1: The first three terms are considered that is $m = 3$. The three eigen values of matrix B may be obtained as $3.4129,\ 24.1292$ and 63.9000. The matrix of the Eigen vectors may be obtained as $V = \begin{pmatrix} -0.9976 & -0.2776 & 0.0794 \\ 0.0689 & 0.9552 & -0.2304 \\ 0.0030 & -0.1021 & 0.9699 \end{pmatrix}$, and the first three coefficients of equation (23) are obtained as follows:

$$c_1(T) = [-1.3887\exp(-3.4129T) + 0.1316\exp(-24.1292T) - 0.0193\exp(-63.9000T)$$
$$-0.0418(1-\exp(-3.4129T)) + 0.0006(1-\exp(-24.1292T))]$$
$$c_2(T) = [0.0960\exp(-3.4129T) - 0.4530\exp(-24.1292T) + 0.0560\exp(-63.9000T) +$$
$$-0.0029(1-\exp(-3.4129T)) - 0.0019(1-\exp(-24.1292T)) + 0.0001\exp(1-63.9000T)]$$
$$c_3(T) = [0.0042\exp(-3.4129T) + 0.0484\exp(-24.1292T) - 0.2359\exp(-63.9000T) +$$
$$+0.0001(1-\exp(-3.4129T)) + 0.0002(1-\exp(-24.1292T)) - 0.0004(1-\exp(-63.9000T))]$$

Using above three coefficients in equation (23) and using the transformation in equation (11) the desired solution in non-dimensional variables may be written as

$$C^*(X,T) = \exp(-X/2)\left[\frac{c_1(T)\sin(1.165561X)}{0.5870} + \frac{c_2(T)\sin(4.604217X)}{0.6988} + \frac{c_3(T)\sin(7.789884X)}{0.7042}\right]$$
$$+(1+0.05X) \qquad (28)$$

Case 2: $m = 5$. The eigen values of matrix B are obtained as $207.7989,\ 127.6455,\ 3.4234,\ 24.1673,\ 65.5230$ and matrix of the Eigen vectors as follows:

$$V = \begin{pmatrix} -0.0252 & 0.0486 & 0.9976 & 0.2781 & -0.0850 \\ 0.0352 & -0.0518 & -0.0691 & -0.9551 & 0.2275 \\ -0.0400 & 0.2219 & -0.0037 & 0.1022 & -0.9640 \\ 0.2293 & -0.9665 & -0.0032 & 0.0014 & 0.1081 \\ -0.9716 & 0.1076 & -0.0007 & 0.0070 & -0.0025 \end{pmatrix}.$$

The first five coefficients of equation (23) may be obtained as follows:

$$c_1(T) = -0.0033\exp(-207.7992T) + 0.0098\exp(-127.6455T) - 1.3914\exp(-3.4233T)$$
$$+0.1330\exp(-24.4673T) - 0.0244\exp(-65.5231T) - 0.0417(1-\exp(-3.4233T))$$
$$+0.0006(1-\exp(-24.4673T))]$$

$$c_2(T) = [0.0046\exp(-207.7992T) - 0.0104\exp(-127.6455T) + 0.0964\exp(-3.4233T)$$
$$-0.4568\exp(-24.4673T) + 0.0653\exp(-65.5231T)$$
$$+0.0029(1-\exp(-3.4233T)) - 0.0019(1-\exp(-24.4673T)) + 0.0001(1-\exp(-65.5231T))]$$

$$c_3(T) = [-0.0052\exp(-207.7992T) + 0.0445\exp(-127.6455T) + 0.0052\exp(-3.4233T)$$
$$+0.0489\exp(-24.4673T) - 0.2767\exp(-65.5231T) + 0.0002(1-\exp(-3.4233T))$$
$$+0.0002(1-\exp(-24.4673T)) - 0.004(1-\exp(-65.5231T))]$$

$$c_4(T) = [0.0298\exp(-207.7992T) - 0.1940\exp(-127.6455T) + 0.0045\exp(-3.4233T)$$
$$+0.0007\exp(-24.4673T) + 0.0310\exp(-65.5231T) - 0.0002(1-\exp(-127.6455T))]$$

$$c_5(T) = [-0.1261\exp(-207.7992T) + 0.0210\exp(-127.6455T) - 0.0010\exp(-3.4233T)$$
$$+0.0033\exp(-24.4673T) - 0.0007\exp(-65.5231T) - 0.0001(1-\exp(-207.7992T))]$$

So the solution in non-dimensional variables may be written

$$C^*(X,T) = \exp(-X/2)\left[\frac{c_1(T)\sin(1.165561X)}{0.5870} + \frac{c_2(T)\sin(4.604217X)}{0.6988} + \frac{c_3(T)\sin(7.789884X)}{0.7042}\right.$$
$$\left. + \frac{c_4(T)\sin(10.949944X)}{0.7056} + \frac{c_5(T)\sin(14.101726X)}{0.7062}\right] + (1+0.05X) \qquad (29)$$

Case3: $m = 8$ The eight eigen values of matrix B may be obtained as 579.5089, 438.2884, 313.9846, 210.5699, 3.4258, 24.1741, 64.6024, 127.7221 and matrix of the Eigen vectors as

$$V = \begin{pmatrix} 0.0098 & -0.0131 & -0.0194 & 0.0272 & -0.9976 & 0.2780 & -0.0857 & -0.0490 \\ -0.0085 & 0.0148 & 0.0177 & -0.0366 & 0.0691 & -0.9551 & 0.2276 & 0.0523 \\ 0.0129 & -0.0144 & -0.0328 & 0.0440 & 0.0038 & 0.1024 & -0.9639 & -0.2222 \\ -0.0117 & 0.0309 & 0.0404 & -0.2242 & 0.0032 & 0.0015 & 0.1084 & 0.9664 \\ 0.0296 & -0.0391 & -0.2292 & 0.9671 & 0.0008 & 0.0071 & -0.0007 & -0.1067 \\ -0.0368 & 0.2349 & 0.9669 & -0.1020 & 0.0007 & 0.0009 & 0.0093 & 0.0022 \\ 0.2476 & -0.9663 & -0.0953 & 0.0029 & 0.0003 & 0.0018 & 0.0006 & -0.0105 \\ -0.9675 & 0.0898 & 0.0051 & -0.0113 & 0.0003 & 0.0004 & 0.0025 & 0.0001 \end{pmatrix}$$

The first eight coefficients of the series (23) may be obtained as

$$c_1(T) = [0.0007\exp(-579.5089T) - 0.0014\exp(-438.2884T) + 0.0025\exp(-313.9846T)$$
$$+0.0043\exp(-210.5699T) - 1.3921\exp(-3.34258T) + 0.1331\exp(-24.1741T))$$
$$-0.0249\exp(-65.6024T) + 0.0100\exp(-127.7221T) - 0.0417(1-\exp(-210.5699T))$$
$$+0.0006(1-\exp(-24.1741T))]$$

$$c_2(T) = [-0.0006\exp(-579.5089T) + 0.0016\exp(-438.2884T) - 0.0023\exp(-313.9846T)$$
$$+0.0058\exp(-210.5699T) + 0.0964\exp(-3.34258T) - 0.4573\exp(-24.1741T))$$
$$+0.0660\exp(-65.6024T) - 0.0107\exp(-127.7221T) + 0.0029(1-\exp(-3.4258\,\mathrm{T}))$$
$$-0.0019(1-\exp(-24.1741T)) + 0.0001(1-\exp(-65.6024T))\ ]$$

$$c_3(T) = [+0.0010\exp(-579.5089T) - 0.0015\exp(-438.2884T) - 0.0042\exp(-313.9846T)$$
$$-0.0070\exp(-210.5699T) - 0.0053\exp(-3.34258T)\ 0.0490\exp(-24.1741T))$$
$$-0.2796\exp(-65.6024T) + 0.0455\exp(-127.7221T) + 0.0002(1-\exp(-3.4258T))$$

$$+0.0002(1-\exp(-24.1741T)) \; -0.0004(1-\exp(-65.6024T)) \,]$$

$$\begin{aligned}
c_4(T) = [&-0.009\exp(-579.5089T) +0.0033\exp(-438.2884T) +0.0052\exp(-313.9846T) \\
&+0.0358\exp(-210.5699T) +0.0045\exp(-3.34258T) +0.0007\exp(-24.1741T)) \\
&+0.0314\exp(-65.6024T) -0.1977\exp(-127.7221T) +0.0001(1-\exp(-3.4258T)) \\
&+0.0002(1-\exp(-127.7221T)) \,] \\
c_5(T) = [&0.0022\exp(-579.5089T) -0.0042\exp(-438.2884T) -0.0294\exp(-313.9846T) \\
&-0.1543\exp(-210.5699T) +0.0011\exp(-3.34258T) +0.0034\exp(-24.1741T)) \\
&-0.0027\exp(-65.6024T) +0.0218\exp(-127.7221T) -0.0001(1-\exp(-3.4258T)) \,] \\
c_6(T) = [&-0.0027\exp(-579.5089T) +0.0251\exp(-438.2884T) +0.1240\exp(-313.9846T) \\
&+0.0163\exp(-210.5699T) +0.0010\exp(-3.34258T) +0.0004\exp(-24.1741T)) \\
&+0.0027\exp(-65.6024T) -0.0005\exp(-127.7221T) \,] \\
c_7(T) = [&0.0184\exp(-579.5089T) -0.1034\exp(-438.2884T) -0.0122\exp(-313.9846T) \\
&-0.0005\exp(-210.5699T) +0.0004\exp(-3.34258T) +0.0009\exp(-24.1741T)) \\
&+0.0002\exp(-65.6024T) +0.0021\exp(-127.7221T) \,] \\
c_8(T) = [&-0.0719\exp(-579.5089T) \; 0.0096\exp(-438.2884T) +0.0007\exp(-313.9846T) \\
&+0.0018\exp(-210.5699T) +0.0004\exp(-3.34258T) +0.0002\exp(-24.1741T)) \\
&+0.0007\exp(-65.6024T) \,].
\end{aligned}$$

The analytical solution in non-dimensional variables may be written

$$\begin{aligned}
C^*(X,T) = \exp(-X/2)\Bigg[&\frac{c_1(T)\sin(1.165561X)}{0.5870} + \frac{c_2(T)\sin(4.604217X)}{0.6988} + \frac{c_3(T)\sin(7.789884X)}{0.7042} \\
&+ \frac{c_4(T)\sin(10.949944X)}{0.7056} + \frac{c_5(T)\sin(14.101726X)}{0.7062} + \frac{c_6(T)\sin(17.249783X)}{0.7065} \\
&+ \frac{c_7(T)\sin(20.395842X)}{0.7067} + \frac{c_8(T)\sin(23.540709X)}{0.7068}\Bigg] + (1+0.05X) \qquad (30)
\end{aligned}$$

The analytical solution using the SLP comprising of the ODE in equation (19) and the conditions in equations (17) and (18) for first five terms of the corresponding Fourier series may be obtained as

$$\begin{aligned}
C^*(X,T) = \sqrt{2}\Bigg[&c_1(T)\sin\left(\frac{3\pi}{2}X\right) + c_2(T)\sin\left(\frac{5\pi}{2}X\right) + c_3(T)\sin\left(\frac{7\pi}{2}X\right) + c_4(T)\sin\left(\frac{9\pi}{2}X\right) \\
&+ c_5(T)\sin\left(\frac{11\pi}{2}X\right)\Bigg] + 1 + 0.05X\,, \qquad (31)
\end{aligned}$$

where the coefficients are as follows:

$$\begin{aligned}
c_1(T) = &-0.0001\exp(-313.8687T) - 0.3044\exp(-23.5733T) + 0.0003\exp(-209.9019T) \\
&-0.0003\exp(-127.0565T) +0.0075\exp(-64.9353T) -0.0041(1-\exp(-23.5733T)) \\
&+0.0001(1-\exp(-64.9353T)) \\
c_2(T) = &\;0.0003\exp(-313.8687T) - 0.0003\exp(-23.5733T) - 0.0002\exp(-209.9019T) \\
&+0.0060\exp(-127.0565T) -0.1870\exp(-64.9353T) - 0.0017(1-\exp(-64.9353T))
\end{aligned}$$

$$c_3(T) = -0.0002\exp(-313.8687T) + 0.0003\exp(-23.5733T) + 0.0055\exp(-209.9019T)$$
$$-0.1310\exp(-127.0565T) - 0.0025\exp(-64.9353T) - 0.0006(1-\exp(-127.0565T))$$
$$c_4(T) = 0.0049\exp(-313.8687T) + 0.0002\exp(-23.5733T) - 0.1022\exp(-209.9019T)$$
$$-0.0032\exp(-127.0565T) + 0.0001\exp(-64.9353T) - 0.0002(1-\exp(-209.9019T))$$
$$c_5(T) = -0.0781\exp(-313.8687T) + 0.0001\exp(-23.5733T) - 0.0036\exp(-209.9019T).$$

The analytical solutions $c(x,t)$ are illustrated through the figures. Each figure shows the expected pattern of the pollutant mass transport originating from the uniform continuous point source situated at the origin. The concentration values attenuate with position at a time and increase with time at a particular position. These figures exhibit that the input concentration that is the concentration at the origin of the domain remains uniform with a value 1.0 at each time. Fig. 1 illustrates the convergence of the extended Fourier series in equation (23) to the analytical solutions $c(x,t)$ at $t = 1.0$ and 2.0 (years). The dotted curves are drawn using the solution in equation (28) obtained for three terms ($m = 3$) and solid curves represent the solutions in equations (29) and (30), for five and eight terms respectively. It may be observed that the solid curves at each time are absolutely same for both the solutions. In other words increase in the number of terms from five to eight in the Fourier series does not affect the concentration pattern at all. Even the concentration pattern has slight difference between the solutions in equations (28) and (29) obtained for three and five terms, respectively. That difference too decreases with time. It shows that the analytical solution obtained by GITT using a new differential equation in the SLP is very much convergent with much less number of terms of the extended Fourier series. As a result it has been possible to write analytical expressions. This is not possible using the ODE in equation (19) as part of the SLP, which has been used in almost all the previous works using this method. The analytical solution using equation (19) as part of the SLP for five terms of the corresponding Fourier series is written in equation (31). This solution is demonstrated in fig. 2. As evident from the figure the analytical solution is highly unstable. This unstability vanishes and the solution converges to the expected concentration pattern using large number of terms of the Fourier series. In the previous works this number varies from thirty to hundred. In the paper of Liu et al. (2000) thirty terms are considered while this numer is 50 in the work of Costa et al. (2006). Due to such large number of terms the system of initial value problem may only be solved numerically.

The authenticity of the analytical solutions obtained in the present paper is also checked. The dispersion problem considered by Liu et al. (2000) is solved numerically using the software MATLAB PDEPE Solver. The concentration values obtained through it are found exactly same as those illustrated in that paper. It validates the accuracy of the software used in solving such dispersion problems numerically. The dispersion problem of the present paper is also solved through this software and the concentration values obtained are found matching exactly with the concentration values evaluated from the analytical solution in equation (29) for five terms at $t = 0.5$, 1.0 and 2.0 (years). The comparison is shown in fig. 3. This proves the genuineness of the analytical solutions obtained in this paper using a new SLP.

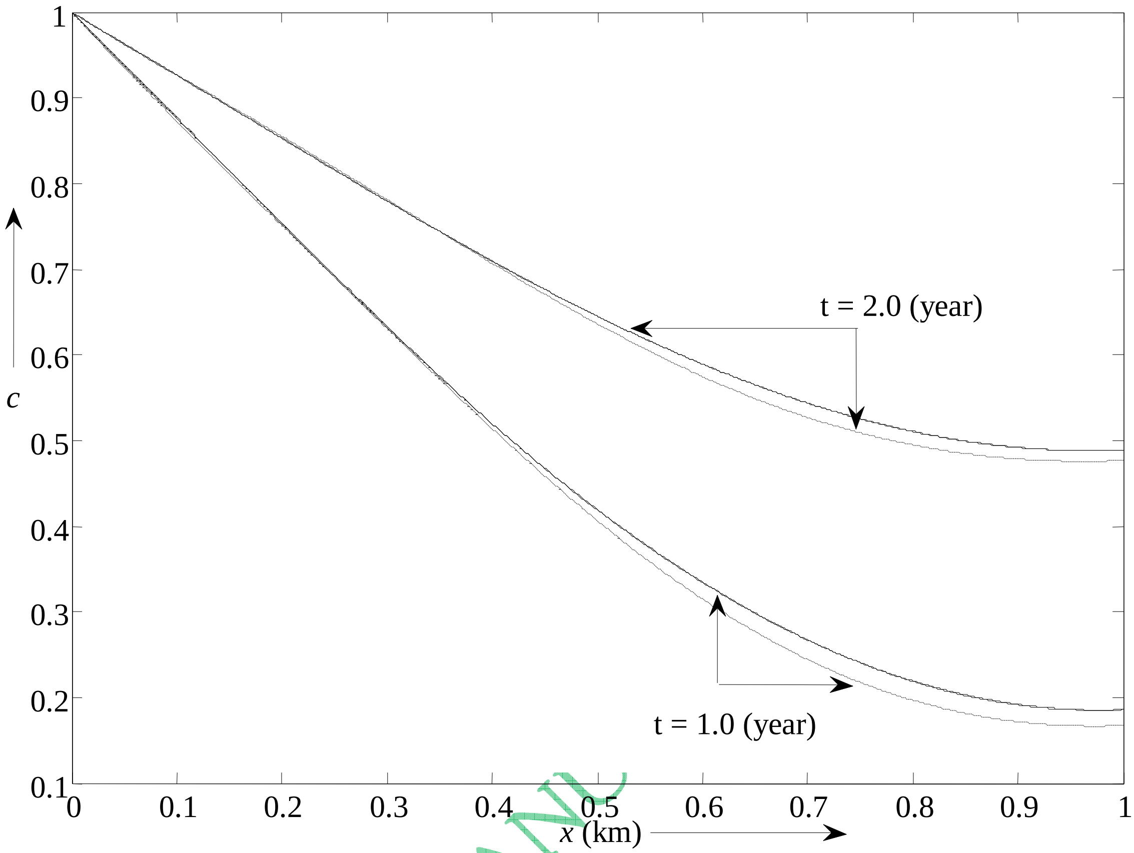

Fig. 1. Concentration values evaluated from equations (28) for three terms, represented by dotted curves and equations (29) and (30) for five and eight terms both coinciding exactly with solid curves.

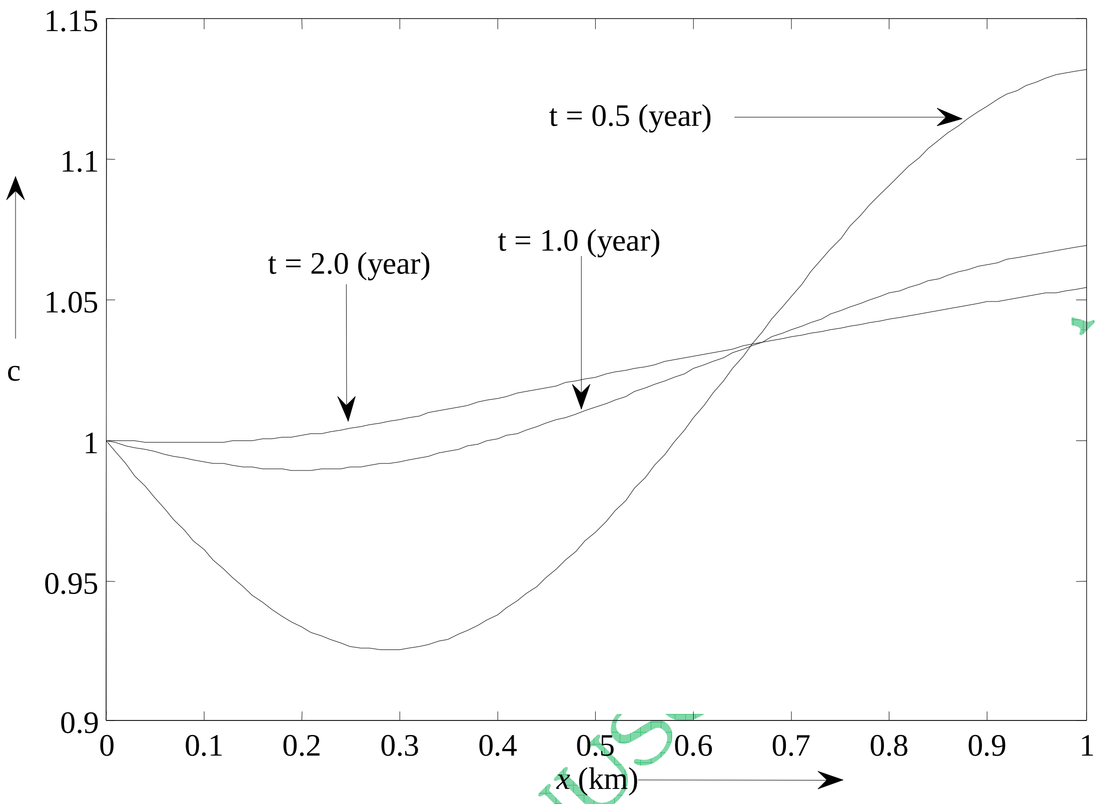

Fig. 2. Concentration values evaluated from equation (31) for five terms obtained through the SLP used in previous works.

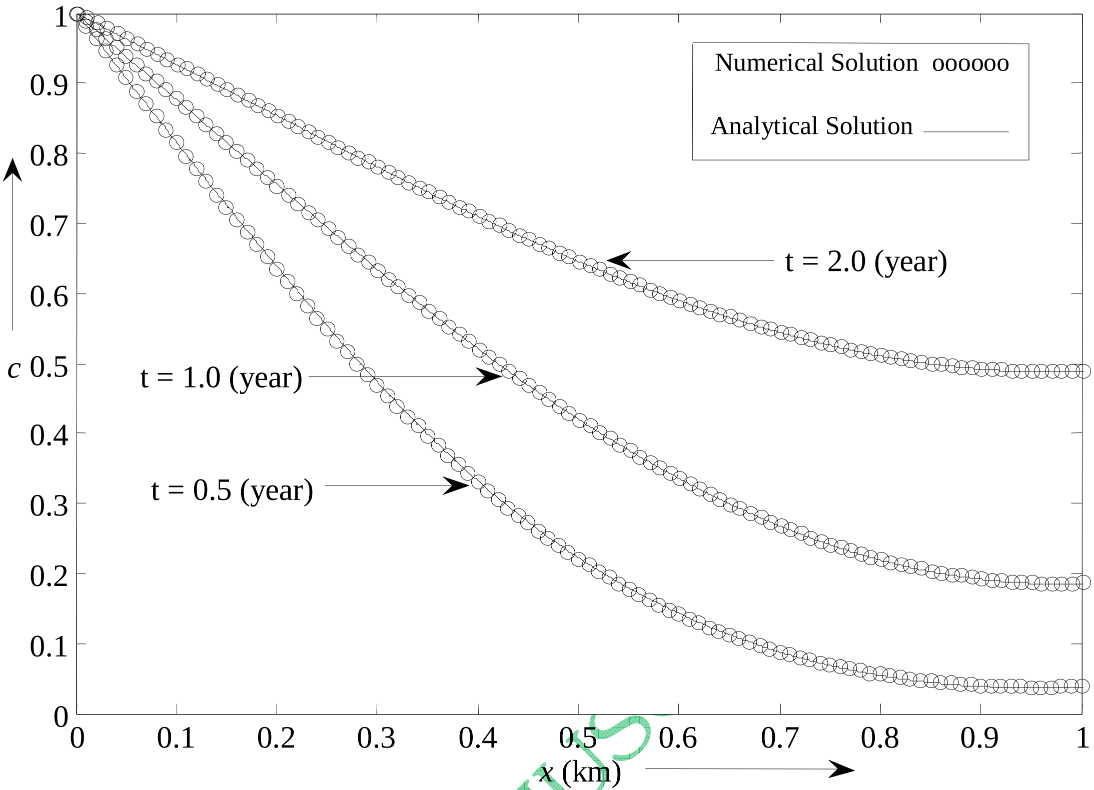

Fig. 3. Comparison of concentration values evaluated from analytical solution in equation (29) with those obtained by solving the problem numerically.

4. Conclusion

In the previous works analytical solutions have not been possible while solving a solute dispersion problem associated with an ADE using the GITT because of the convergence of the extended Fourier series to the desired solution occurring only if the large number of terms like 30, 50 or more terms is considered in the series. The differential equation of the Sturm-Liouville's problem (SLP) is same in all these works. In the present paper a new ODE is used as part of the SLP. As a result the extended Fourier series converges to the desired solution with much less number of terms hence analytical expressions for the concentration are given. The medium is considered heterogeneous which is defined by velocity as a linear non-homogeneous function of position variable. The other coefficient of the ADE, dispersion coefficient is considered proportional to velocity, a case for which analytical solution is not possible by any other integral transform technique. In future works the authors aim to use the same technique to get analytical solutions with more purposeful types of point sources, and expressions describing the heterogeneity.

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